APPENDIX B

LAPLACE TRANSFORMS

The previous sections have reviewed the classical method of solving linear differential equations. We have seen how the transient solution and steady state solution are derived and have developed the concept of operational impedance and impedance to a constant frequency applied excitation function.

These same results can be derived in a greatly simplified fashion through the use of the direct and inverse LaPlace transform which uses one approach for both the steady state and transient solution. LaPlace transform operational calculus is the cornerstone of control system analysis.

Some Basic Theorems of the LaPlace Transform

A function of time \( f(t) \) has a LaPlace transform \( F(s) \) where

\[
F(s) = \int_{0}^{\infty} f(t) \, e^{-st} \, dt
\]  

(B-1)

The value of the LaPlace transform lies in the fact that a differential equation or expression of the variable "\( t \)" transforms into an algebraic equation or expression of the variable "\( s \)." This algebraic expression in turn may be operated upon and converted to a form easily recognized in terms of a time function. The process of obtaining the time function from the transform expression is called taking the inverse LaPlace transformation. Mathematical operations which in the time domain involve convolution, convert to simple algebraic multiplications in the \( s \) domain. A summary of the important theorems governing the use of the LaPlace transform are:

1. \( \mathcal{L} \left[ f(t) \right] = \int_{0}^{\infty} f(t) \, e^{-st} \, dt \)  
   (B-2)

2. The inverse LaPlace transformation \( \mathcal{L}^{-1} \) is defined implicitly by the relation

\[
\mathcal{L}^{-1} \left( \mathcal{L} \left[ f(t) \right] \right) = f(t) \hspace{0.5cm} 0 \leq t
\]  
   (B-3)

3. If the functions \( f(t) \), \( f_1(t) \) and \( f_2(t) \) have \( \mathcal{L} \) transforms \( F(s) \), \( F_1(s) \) and \( F_2(s) \) respectively and "\( a \)" is a constant of a variable which is independent of \( t \) and \( s \), then

\[
\mathcal{L} \left[ a \, f(t) \right] = a \, F(s)
\]  
   (B-4)

and

\[
\mathcal{L} \left[ f_1(t) \pm f_2(t) \right] = F_1(s) \pm F_2(s)
\]  
   (B-5)
Also

\[ \mathcal{L}^{-1} \left[ a \, F(s) \right] = a \, f(t) \quad 0 \leq t \]  

(B-6)

and

\[ \mathcal{L}^{-1} \left[ F_1(s) + F_2(s) \right] = f_1(t) + f_2(t) \quad 0 \leq t \]  

(B-7)

4. If a function \( f(t) \) has the \( \mathcal{L} \) transform \( F(s) \), then

\[ \mathcal{L} \left( \frac{df(t)}{dt} \right) = s \, F(s) - f(0+) \]  

(B-8)

where \( f(0+) \) is the value of \( f(t) \) at \( t = 0^+ \). It is evident then that

\[ \mathcal{L} \left( \frac{d^2f(t)}{dt^2} \right) = s^2 \, F(s) - s \, f(0) - f'(0) \]  

(B-9)

and

\[ \mathcal{L} \left( f^{(n)}(t) \right) = s^n F(s) - \sum_{k=1}^{n} \frac{f^{(k-1)}(0) \, s^{n-k}}{k!} \]  

(B-10)

5. If the function \( f(t) \) has the transform \( F(s) \), its integral \( f^{(-1)}(t) = \int f(t) \, dt = f(t) \) has the transform

\[ \mathcal{L} \left[ \int f(t) \, dt \right] = \frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s} \]  

(B-11)

Similarly,

\[ \mathcal{L} \left[ f^{(-2)}(t) \right] = \frac{F(s)}{s^2} + \frac{f^{(-1)}(0) + f^{(-2)}(0)}{s^2} \]  

(B-12)

and

\[ \mathcal{L} \left[ f^{(-n)}(t) \right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{f^{(-k)}(0)}{s^{n-k+1}} \]  

(B-13)

6. The Laplace transforms of some common functions are as follows:
\[ f(t) \quad 0 \leq t \quad F(s) \]

1 or \( u(t) \)

\[ \frac{1}{s} \]

\( e^{-at} \)

\[ \frac{1}{s + a} \]

\[ \frac{1}{s^2 + \beta^2} \]

\[ \frac{s}{s^2 + \beta^2} \]

\[ \frac{1}{s} \]

\[ \frac{1}{s^n} \]

\[ \frac{1}{s^n} \]

\[ \frac{1}{(s + a)^2} \]

\[ \frac{1}{(s + a)^n} \]

\[ \frac{1}{s} e^{-as} \]

\[ \frac{1}{s} (e^{-as} - e^{-bs}) \]

\[ \text{Unit Impulse} \]

\[ u_1(t) = \lim_{a \to 0} \frac{u(t) - u(t-a)}{a} = 1 \]

The \( \mathcal{L} \) Transformation

We shall now apply Laplace transform methods to the solution of differential equations. Take for instance:

\[ A \frac{d^2y}{dt^2} + B \frac{dy}{dt} + Cy = f(t) \ldots, \quad y \neq y(t) \quad (B-15) \]

in which \( A, B, \) and \( C \) are known constants.
The unknown \( y(t) \) will be called the response function and the known \( f(t) \) will be called the driving function. The initial values of the unknown and its first derivative are \( y(0) \) and \( y'(0) \).

Applying the \( \mathcal{L} \) transformation to both members of equation B-15

\[
\mathcal{L} \left[ A \frac{d^2 y}{dt^2} + B \frac{dy}{dt} + Cy \right] = \mathcal{L} \left[ f(t) \right]
\]  

(B-16)

Calling \( F(s) \) the \( \mathcal{L} \) transform of \( f(t) \) and \( Y(s) \) \( \mathcal{L} \left[ y(t) \right] \), the response transform. Then, using equation B-8 and equation B-9

\[
\mathcal{L} \left[ y'(t) \right] = sY(s) - y(0)
\]

and

\[
\mathcal{L} \left[ y''(t) \right] = s^2 Y(s) - y(0) s - y'(0)
\]

This discloses the way in which the initial conditions \( y(0) \) and \( y'(0) \) are incorporated in the solution during the process of transformation.

Equation B-16 becomes

\[
A \mathcal{L} \left[ \frac{d^2 y}{dt^2} \right] + B \mathcal{L} \left[ \frac{dy}{dt} \right] + C \mathcal{L} \left[ y \right] = \mathcal{L} f(t)
\]

\[
A \left[ s^2 Y(s) - y(0) s - y'(0) \right] + B \left[ sY(s) - y(0) \right] + CY(s) = F(s)
\]

or

\[
(A s^2 + B s + C) Y(s) = F(s) + y(0)(A s + B) + y'(0) A
\]  

(B-17)

Equation B-17 is called a transform equation. The polynomial coefficient of \( Y(s) \) - in this case \( (A s^2 + B s + C) \) - is called the characteristic function since it completely characterizes the physical system described by the differential equation. Note that this is identical with the system characteristic equation derived in Appendix A, except for the variable "\( s \)" instead of the operator "\( p \)." The equation formed by setting it to zero is called the characteristic equation of the system. Solving equation B-17 algebraically,

\[
Y(s) = \frac{1}{A s^2 + B s + C} \left[ F(s) + y(0)(A s + B) + y'(0) A \right]
\]  

(B-18)

This algebraic equation has a form which will be found typical of all transform solutions, viz:

Response transform = System function \times Excitation function

The system function in this example is the reciprocal of the characteristic function, but in general it will be a fraction of which the characteristic function is the denominator. It incorporates in one function all the essential knowledge regarding the physical system.
The excitation function includes the driving transform and the initial conditions. It contains all the essential specifications of the excitations applied to the system.

When the form of the driving function \( f(t) \) is specified, the algebraic form of \( Y(s) \) can be determined and

\[
y(t) = \mathcal{L}^{-1} \left[ Y(s) \right] = \mathcal{L}^{-1} \left[ \frac{F(s) + y(0)(A s + R) + y'(0)A}{A s^2 + B s + C} \right]
\]

(B-19)

If \( Y(s) \) were an algebraic function of the form of any one of the various transforms listed so far, the inverse could be written immediately by reference to the table. But since \( Y(s) \) is a more complicated function than listed, such a direct method of determining the inverse transform fails.

This difficulty may be surmounted by resolving the function into a sum of simpler components whose inverse transforms are readily recognized.

\( \mathcal{L}^{-1} \) Transformation

Consider the general rational algebraic fraction

\[
F(s) = \frac{A(s)}{B(s)} = \frac{a_p s^p + a_{p-1} s^{p-1} + \cdots + a_1 + a_0}{s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0}
\]

(B-20)

where \( p \leq q \)

By solving for the roots of the equation \( B(s) = 0 \), and calling these \( s_1, s_2, \ldots s_q \), the fraction may be expressed as

\[
F(s) = \frac{A(s)}{B(s)} = \frac{A(s)}{(s - s_1)(s - s_2)(s - s_3) \cdots (s - s_q)}
\]

(B-21)

and the above may in turn be written as a sum of partial fractions, each partial fraction having for its denominator one of the factors of \( B(s) \).

There will be "q" of these partial fractions.

i.e.,

\[
\frac{A(s)}{B(s)} = \frac{K_1}{(s - s_1)} + \frac{K_2}{(s - s_2)} + \frac{K_3}{(s - s_3)} + \cdots \frac{K_q}{(s - s_q)}
\]

(B-22)

To evaluate the typical coefficient \( K_k \), multiply both members of equation B-22 by \( (s - s_k) \) obtaining

\[
\frac{(s - s_k)A(s)}{B(s)} = K_1 \frac{(s - s_k)}{(s - s_1)} + K_2 \frac{(s - s_k)}{(s - s_2)} + \cdots + K_q \frac{(s - s_k)}{(s - s_q)}
\]

(B-23)

In the fraction forming the left member of equation B-23, \( (s - s_k) \) is a factor of both numerator and denominator and should be divided out. Then letting \( s = s_k \), this left member becomes a numerator, and in the right member all terms except \( K_k \) become zero.
\[ K_k = \left[ \frac{(s - s_k)A(s)}{B(s)} \right] s = s_k \]  
\[ s_k \]

\[ A(s_k) = (s_k - s_1)(s_k - s_2)\ldots(s_k - s_{k-1})(s_k - s_{k+1})\ldots(s_k - s_q) \]

But
\[ (s_k - s_1)(s_k - s_2)\ldots(s_k - s_{k-1})(s_k - s_{k+1})\ldots(s_k - s_q) \]

\[ = \left[ \frac{d}{ds} B(s) \right]_s = B'(s_k) = \Delta \]  
\[ s = s_k \]

so equation B-24 can be written

\[ \frac{A(s)}{B(s)} = \sum_{k=1}^{g} \frac{A(s_k)}{B'(s_k)} - \frac{1}{s - s_k} \]  
\[ \text{(B-26)} \]

The actual problem of inverse transformation is now a simple one.

\[ L^{-1} \left[ \frac{1}{s - s_k} \right] = e^{s_k t} \]

The above holds for \( \frac{A(s)}{B(s)} \) having first order poles only; i.e., the roots of \( B(s) \) being

\[ (s + s_1)^n (s + s_2)^m (s + s_3)^\ell \ldots \]

where \( n, m, \ell = 1 \)

and \( s_1 \neq s_2 \neq s_3 \ldots \)

Example:

Find the

\[ L^{-1} \left[ \frac{a_1 s + a_0}{(s + \alpha_1)(s + \alpha_2)(s + \alpha_3)} \right] \]

in which \( \alpha_1, \alpha_2, \text{ and } \alpha_3 \) are real numbers, all different.

\[ L^{-1} \left[ \frac{a_1 s + a_0}{(s + \alpha_1)(s + \alpha_2)(s + \alpha_3)} \right] = k_1 e^{-\alpha_1 t} + k_2 e^{-\alpha_2 t} + k_3 e^{-\alpha_3 t} \]

in which
\[ K_1 = \left[ \frac{s + a_0}{s + a_1}(s + a_2) \right]_{s = -\alpha_1} = \frac{-a_1 a_2 + a_0}{(-a_1 + a_2)(-a_1 + a_3)} \]

\[ K_2 = \left[ \frac{s + a_0}{s + a_1}(s + a_2) \right]_{s = -\alpha_2} = \frac{-a_1 a_2 + a_0}{(-a_2 + a_1)(-a_2 + a_3)} \]

\[ K_3 = \left[ \frac{s + a_0}{s + a_1}(s + a_2) \right]_{s = -\alpha_3} = \frac{-a_1 a_2 + a_0}{(-a_3 + a_1)(-a_3 + a_2)} \]

Special case: One pole lies at the Origin.

In \( \frac{A(s)}{B(s)} \) of equation B-21 let \( s_1 = 0 \), then

\[ \frac{A(s)}{B(s)} = \frac{A(s)}{s(s - s_2)(s - s_3)\ldots(s - s_q)} = \frac{A(s)}{s B_1(s)} \]

where \( B_1(s) = \frac{B(s)}{s} \)

The form above occurs frequently. It arises, for example, when the excitation function
is a constant step and the system function does not have a pole or a zero at \( s = 0 \).

The final result can be shown to be

\[ \mathcal{L}^{-1}\left[ \frac{A(s)}{s B_1(s)} \right] = \frac{A(0)}{B_1(0)} + \sum_{k=2}^{q} \frac{A(s_k)}{s_k B_1'(s_k)} e^{s_k t} \quad \text{(B-27)} \]

Example:

Find

\[ \mathcal{L}^{-1}\left\{ \frac{s + a_0}{s((s + \alpha)^2 + \beta^2)} \right\} \]

Here

\[ A(s) = (a_1 s + a_0) \]

\[ B_1(s) = \left[ (s + \alpha)^2 + \beta^2 \right] \]

\( B_1'(s) = 2(s + \alpha) \) and \( s_2, s_3 = -\alpha \pm j\beta \) and \( s_1 = 0 \)

Using equation B-27,
\[
L^{-1} \left( \frac{a_1 s + a_0}{s \left( (s+\alpha)^2 + \beta^2 \right)} \right)
\]

\[= \frac{A(\alpha)}{B(\alpha)} + K_2 e^{-(\alpha + j\beta)t} + K_3 e^{-(\alpha - j\beta)t}\]

where
\[
K_2 = \left[ \frac{a_1 s + a_0}{2s(s+\alpha)} \right]_{s = -\alpha + j\beta}
\]
\[= \frac{a_0 - a_1 \alpha + j a_1 \beta}{2\beta(\alpha + j\beta)}
\]
\[= \frac{1}{2\beta \beta_0} \left[ (a_0 - a_1 \alpha)^2 + a_1^2 \beta^2 \right]^{1/2} e^{j(\phi - \frac{\pi}{2})}
\]

where
\[
\beta_0^2 = \alpha^2 + \beta^2
\]

and
\[
\phi = \left[ \tan^{-1} \frac{a_1 \beta}{a_0 - a_1 \alpha} - \tan^{-1} \frac{\beta}{\alpha} \right]
\]

Similarly
\[
K_3 = \left[ \frac{a_1 s + a_0}{2s(s+\alpha)} \right]_{s = -\alpha - j\beta}
\]
\[= \bar{K}_2 \text{ (conjugate of } K_2)\]

Coefficients \(K_2\) and \(K_3\) are conjugate complex numbers.

The final result can be written

\[
L^{-1} \left[ \frac{a_1 s + a_0}{s \left( (s+\alpha)^2 + \beta^2 \right)} \right]
\]

\[= \frac{a_0}{\beta_0^2} + \frac{1}{\beta \beta_0} \left[ (a_0 - a_1 \alpha)^2 + a_1^2 \beta^2 \right]^{1/2} e^{-\alpha t} \sin(\beta t + \phi) \quad (B-28)
\]

A convenient rule to remember in obtaining the \(L^{-1}\) of a function where one pair of roots are \([s + \alpha)^2 + \beta^2]\) is as follows:
The time function corresponding to the roots \((s + a)^2 + \beta^2\) in

\[
\frac{A(s)}{C(s) ((s + a)^2 + \beta^2)} \quad \text{is} \quad Ke^{-at} \sin (\omega t + \phi) \quad (B-29)
\]

where

\[
K = \left| \frac{A(-a + j\beta)}{B} \right| C(-a + j\beta) \quad (B-30)
\]

and

\[
\phi = \text{angle of } A(-a + j\beta) \text{ minus angle of } C(-a + j\beta) \quad (B-31)
\]

Similarly for a function where one pair of roots are \((s^2 + \omega^2)\) the time function component corresponding to these roots in the function

\[
\frac{A(s)}{C(s)(s^2 + \omega^2)} \quad (B-32)
\]

can be obtained as

\[
K \sin (\omega t + \phi) \quad (B-33)
\]

where

\[
K = \left| \frac{A(j\omega)}{\omega C(j\omega)} \right| \quad (B-34)
\]

and

\[
\phi = \text{angle of } A(j\omega) \text{ minus angle of } C(j\omega) \quad (B-35)
\]

**Multiple Order Poles**

Consider the function \(F(s)\) which has poles of higher order. \((s_1 \text{ occurs } m_1 \text{ times, } s_2 \text{ occurs } m_2 \text{ times, etc.})\)

\[
F(s) = \frac{A(s)}{B(s)} = \frac{A(s)}{(s - s_1)^{m_1} (s - s_2)^{m_2} \ldots (s - s_n)^{m_n}} \quad (B-36)
\]

The fraction \(\frac{A(s)}{B(s)}\) may be resolved into a sum of partial fractions. For each pole \(s_k\) of multiplicity \(m_k\) there are \(m_k\) partial fractions

\[
\frac{K_{k1}}{(s - s_k)^{m_k}}, \quad \frac{K_{k2}}{(s - s_k)^{m_k-1}}, \quad \ldots \quad \frac{K_{km_k}}{(s - s_k)}
\]

in which the \(K\)'s are constants yet to be determined.
Thus the expansion of $A(s)/B(s)$ is

$$
\frac{A(s)}{B(s)} = \frac{K_{11}}{s - s_1}^m_1 + \frac{K_{12}}{(s - s_1)^{m_1-1}} + \cdots + \frac{K_{1j}}{(s - s_1)^{m_1-j+1}} + \cdots + \frac{K_{1m_1}}{(s - s_1)^{m_1-1}}
$$

$$
+ \cdots + \frac{K_{k1}}{s - s_k}^m_k + \frac{K_{k2}}{(s - s_k)^{m_k-1}} + \cdots + \frac{K_{kj}}{(s - s_k)^{m_k-j+1}} + \frac{K_{km_k}}{s - s_k}
$$

$$
+ \cdots
$$

To evaluate the $K_k$ coefficients, first multiply both members of the equation above by $(s - s_k)^m_k$ obtaining

$$
\frac{(s - s_k)^m_k A(s)}{B(s)} = K_{k1} + K_{k2}(s - s_k) + K_{k3}(s - s_k)^2 + \cdots + K_{km_k}(s - s_k)^{m_k-1}
$$

$$
+ (s - s_k)^m_k \left[ \frac{K_{11}}{s - s_1}^m_1 + \cdots + \frac{K_{nm_n}}{s - s_n} \right]
$$

In the left member $(s - s_k)^m_k$ cancels out with that factor which is also a part of $B(s)$. Letting $s = s_k$, this left member becomes a number which should correspond to $K_{k1}$ of the right hand side since all other terms would be zero.

In order to obtain the other coefficients, we note that by differentiating both sides with respect to $s$, the following expression results:

$$
\frac{d}{ds} (s - s_k)^m_k \frac{A(s)}{B(s)} = K_{k2} + 2K_{k3}(s - s_k) + \cdots + (M_k - 1)K_{k_{m_k}}(s - s_k)^{m_k-2}
$$

$$
+ \frac{d}{ds} (s - s_k) \left[ \frac{K_{11}}{s - s_1}^m_1 + \cdots + \frac{K_{nm_n}}{s - s_n} \right]
$$

Letting $s = s_k$, we note that $K_{k2}$ is equal to the number resulting from the evaluation of
\[
\begin{align*}
\left[ \frac{d}{ds} (s - s_k)^m \frac{A(s)}{B(s)} \right]_{s = s_k} & \quad (B-37) \\
\text{Similarly for the other terms} & \\
K_{k3} &= \frac{1}{2!} \frac{d^2}{ds^2} (s - s_k)^m \frac{A(s)}{B(s)} \quad (B-38) \\
&= s = s_k \\
\text{and} & \\
K_{k4} &= \frac{1}{3!} \frac{d^3}{ds^3} (s - s_k)^m \frac{A(s)}{B(s)} \quad (B-39) \\
&= s = s_k
\end{align*}
\]

**Example:**

Find

\[
\mathcal{L}^{-1} \left[ \frac{a_2 s^2 + a_1 s + a_0}{(s+\alpha)^3 s^2} \right] \quad (B-40)
\]

\[
\mathcal{L}^{-1} \left[ \frac{a_2}{(s+\alpha)^3 s^2} \right] = \mathcal{L}^{-1} \left[ \frac{K_{11}}{s+\alpha} \frac{K_{11}}{(s+\alpha)^2} + \frac{K_{12}}{(s+\alpha)^2} + \frac{K_{13}}{s^2} + \frac{K_{21}}{s} \right] \quad (B-41)
\]

\[
= \left( \frac{K_{11}}{2!} \right) t^2 + K_{12} t + K_{13} e^{-\alpha t} + K_{21} t + K_{22} \quad (B-42)
\]

where

\[
K_{11} = \left[ \frac{a_2 s^2 + a_1 s + a_0}{s^2} \right]_{s = -\alpha} = \frac{a_2 \alpha^2 - a_1 \alpha + a_0}{\alpha^2} \quad (B-43)
\]

\[
K_{12} = \left[ \frac{d}{ds} \frac{a_2 s^2 + a_1 s + a_0}{s^2} \right]_{s = -\alpha} = \frac{-a_1 \alpha + 2a_0}{3} \quad (B-44)
\]

\[
K_{13} = \frac{1}{2!} \left[ \frac{d^2}{ds^2} \frac{a_2 s^2 + a_1 s + a_0}{s^2} \right]_{s = -\alpha} = \frac{-a_1 \alpha + 2a_0}{\alpha^4} \quad (B-45)
\]
Let us complete this section by taking the same examples as in Appendix A.

Consider the circuit of Fig. B-1.

The differential equation for condition after closing of the switch at \( t = 0 \) is

\[ E = iR + L \frac{di}{dt} \]  
(B-48)

Taking the \( \mathcal{L} \) transform of both sides of equation B-48

\[ E(s) = Ri(s) + Ls(i(s) - i(0)) \]  
(B-49)

where \( E(s) \) denotes \( \mathcal{L}[E(t)] \)
and \( i(s) \) denotes \( \mathcal{L}[i(t)] \)
and \( i(0) \) = initial condition of \( i \) at \( t = 0 \)

Since \( E \) is a constant, its \( \mathcal{L} \) transform is \( E/S \) (see table of transforms equation B-14). Also for this case \( i(0) = 0 \).

Hence equation B-49 becomes

\[ \frac{E}{S} = i(s) [R + Ls] \]  
(B-50)

Solving for

\[ i(s) = \frac{E}{S[R + Ls]} \]  
(B-51)

Equation B-51 is the \( \mathcal{L} \) transform solution of the current.

To obtain the time domain solution of current we must perform the inverse transform of equation B-51.
i.e.,

\[ i(t) = L^{-1} i(s) = L^{-1} \left( \frac{E}{Ls(s + \frac{R}{L})} \right) \]  \hspace{1cm} (B-52)

Using the rules of partial fraction expansion (equation B-22 to equation B-24).

\[ i(t) = L^{-1} \left[ \frac{K_1}{s} + \frac{K_2}{s + \frac{R}{L}} \right] \]  \hspace{1cm} (B-53)

where

\[ K_1 = \left. \frac{E}{(R + Ls)} \right|_{s=0} = \frac{E}{R} \]

\[ K_2 = \left. \frac{E}{Ls} \right|_{s=-\frac{R}{L}} = -\frac{E}{R} \]

Using the LaPlace transform tables equation B-14 to obtain the inverse of equation B-53.

\[ i(t) = K_1 + K_2 e^{-\frac{R}{L} t} \]

\[ = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L} t} \]  \hspace{1cm} (B-54)

which is the familiar form of the exponential rise in current in the inductive circuit of Fig. B-1.

Of particular interest is the exponential term \( e^{-\frac{R}{L} t} \) which reveals the decay of the transient component.

The coefficient \( L/R \) has the dimensions of seconds and is known as the "time constant" of the circuit. This time constant is defined as the time in seconds for the transient term to be reduced to \( e^{-1} = 0.359 \) of its initial value. Another useful interpretation of the time constant is the time that would be required for the transient to disappear completely if its rate continued at its initial value. (Fig. B-2).
Take now the case treated in Appendix A of the RLC circuit with the sinusoidal excitation voltage.

Again the circuit voltage drop equation is

\[ R i + L \frac{d i}{d t} + \frac{1}{C} \int i d t = E \cos \omega t \]  

(B-55)

Taking the \( \mathcal{L} \) transform of both sides of equation B-55

\[ (R+iLs) \cdot i(s) - Li(0) + \frac{V_{co}}{s} = \frac{E_{s}}{s^{2}+\beta^{2}} \]  

(B-56)

where \( i(0) = \) initial current at \( t = 0 \)
and \( V_{co} = \) initial voltage across the capacitor \( \frac{1}{C} \int i d t \) \( \bigg|_{t=0} \)

For the case where these initial conditions are zero, equation B-56 can be expressed as

\[ i(s) = \frac{1}{(R+iLs) + \frac{1}{Cs}} \frac{E_{s}}{s^{2}+\beta^{2}} \]  

(B-57)

Note that equation B-57 is in the form
Response transform $i(s) = \left[ \frac{1}{\text{System function}} \right] \left[ \frac{1}{R + Ls + \frac{1}{Cs}} \right] \times \left[ \frac{Es}{(s^2 + \omega^2)} \right]

Expressing equation B-57 in terms of poles and zeros

$$i(s) = \frac{EC \cdot s^2}{(1 + RCs + LCs^2) \cdot (s^2 + \omega^2)} = \frac{EC \cdot s^2}{LC(s + R/2L - 1/2 \sqrt{(R/L)^2 - 4 \cdot LC})(s + R/2L + 1/2 \sqrt{(R/L)^2 - 4 \cdot LC})(s^2 + \omega^2)} \quad \text{(B-58)}$$

where the system poles are the roots of $(1 + RCs + LCs^2)$ which are the same as the roots of the characteristic equation A-26 $(1 + RCp + LCp^2)$ in Appendix A.

The time expression for $i(t)$ is obtained by taking the inverse of equation B-58 using the rules in equations B-29 to B-35 and expressing equation B-58 as

$$i(s) = \frac{EC \cdot s^2}{LC[(s + \alpha)^2 + \beta^2] \cdot (s^2 + \omega^2)}$$

$$i(t) = K_1 e^{-\alpha t} \sin (\beta t + \phi_1) + K_2 \sin (\omega t + \phi_2) \quad \text{(B-59)}$$

where

$$K_1 = \frac{EC \cdot s^2}{LC \cdot \beta \cdot \frac{\alpha^2 - 2ja\beta - \beta^2}{\alpha^2 - 2ja\beta + \omega^2}} \quad \text{at} \quad s = -\alpha + j\beta$$

$$= \frac{EC \left[ \frac{\alpha^2 - 2ja\beta - \beta^2}{\alpha^2 - 2ja\beta + \omega^2} \right]}{LC \cdot \beta}$$

$$= \frac{EC \left[ (\alpha^2 - \beta^2)^2 + 4a^2 \beta^2 \right]^{1/2}}{LC \cdot \beta \left[ (\alpha^2 - \beta^2 + \omega^2)^2 + 4a^2 \beta^2 \right]^{1/2}}$$

and
\[ \phi_1 = \tan^{-1} \frac{-2a \theta}{a^2 - \beta^2} - \tan^{-1} \frac{-2a \theta}{a^2 - \beta^2 + \omega^2} \]

\[ K_2 = \frac{EC \frac{s^2}{(s + a)^2 + \beta^2}}{\omega LC} \bigg|_{s = j\omega} \]

See B-32 to B-35

\[ = \frac{EC \omega^2}{\omega LC \left[ (a^2 + \beta^2 - \omega^2)^2 + 4a^2 \omega^2 \right]^{1/2}} \]

and

\[ \phi_2 = \pi - \tan^{-1} \frac{2a \omega}{a^2 + \beta^2 - \omega^2} \]

Note that equation B-59 has the total solution, steady state (second term) and transient (first term) obtained by a straightforward routine use of the direct and inverse Laplace transform.