APPENDIX A

DYNAMIC SYSTEMS, DIFFERENTIAL EQUATIONS - TRANSIENT
AND STEADY STATE SOLUTIONS - OPERATIONAL IMPEDANCE

The study of "control and dynamics" requires the use of certain mathematical tools and
techniques which have become an essential part of the technology of control. These tools are all
related to methods of solution and analysis of systems described by differential equations. It is
not the intent here to go through a detailed theoretical development of the pertinent mathematics
that form the basis for the various analysis tools. There are numerous texts that may be refer-
enced for this purpose, some of which are in the reference list. The treatment in these appen-
dices will be in the form of a brief review of some basic techniques to supplement and support the
material in the main text on "Generation Dynamics and Control."

Dynamic Systems

The behavior of dynamic systems is expressed by differential equations relating the sys-
tems' variables. In many cases these equations turn out to be or can be approximated by linear
differential equations. When this is the case, classical or closed form solutions can be obtained.

For the general case of non-linear differential equations, solutions must be sought
through the use of simulation by analog computation methods or by numerical integration techniques
carried out on digital computers.

Although any problem can be solved by these simulation methods, the insight that can be
derived from linear system analysis is invaluable as a guide to control system design and perform-
ance evaluations.

System Differential Equations

Dynamic systems can be thermal, mechanical, electrical or a combination of all these. In
order to stay on familiar ground we will illustrate with an electrical example and limit the dis-
cussion to linear differential equations.

Consider the circuit in Fig. A-1.

![Circuit Diagram]

FIGURE A-1

The differential equation is

\[ E = iR + L \frac{di}{dt} \]  

(A-1)
By separating variables, equation A-1 can be put in the form of A-2

$$L \int \frac{df}{E-R_1} = \int dt$$  \hspace{1cm} (A-2)

Integration of equation A-2 yields

$$-\frac{L}{R} \ln (E-R_1) = t + C_1$$  \hspace{1cm} (A-3)

where $C_1$ is the constant of integration.

Equation A-3 may also be expressed in exponential form as

$$i = \frac{E}{R} + C_2 e^{-(R/L)t}$$  \hspace{1cm} (A-4)

where $C_2$ is derived from constants of integration which in turn are determined from initial conditions in energy storage elements. The current in inductance $L$ at time $t = 0$ before the switch is closed is $i_0 = 0$.

Substitution of $i = 0$ at $t = 0$ in equation A-4 yields

$$C_2 = -\frac{E}{R}$$  \hspace{1cm} (A-5)

and equation A-4 can be written as

$$i = \frac{E}{R} \left[ 1 - e^{-(R/L)t} \right]$$  \hspace{1cm} (A-6)

plotted in Fig. A-2 as function of time.

![Figure A-2](image)

This classical solution can be recognized as containing two components:

1. The steady state component
\[ i_s = \frac{E}{R} \]  \hspace{1cm} (A-7)

which has the same form as the applied voltage.

(2) The transient component

\[ i_t = -\frac{E}{R} e^{-(R/L)t} \]  \hspace{1cm} (A-8)

which decays exponentially to zero.

An alternate method of solution for the current in the circuit of Fig. A-1 is to solve separately for the steady state and transient components as follows:

Let

\[ i = i_s + i_t \]  \hspace{1cm} (A-9)

Substituting equation A-9 in equation A-1

\[ E = i_s R + L \frac{di_s}{dt} + R i_t + L \frac{di_t}{dt} \]  \hspace{1cm} (A-10)

Since \( E \) is constant \( \frac{di}{dt} = 0 \).

By definition also, \( i_t \) and \( \frac{di}{dt} \) = 0 in the steady state.

Hence

\[ i_s = \frac{E}{R} \]  \hspace{1cm} (A-11)

Substituting equation A-11 into equation A-10 yields the relation from which the transient component may be solved, i.e.:

\[ R i_t + L \frac{di_t}{dt} = 0 \]  \hspace{1cm} (A-12)

By definition of the transient component it is exponential in nature, and one can express it as

\[ i_t = I_t e^{pt} \]  \hspace{1cm} (A-13)

Substituting equation A-13 in equation A-12

\[ (R + pL) I_t e^{pt} = 0 \]  \hspace{1cm} (A-14)

From equation A-14 the value of \( p \) is determined as

\[ p = -\frac{R}{L} \]  \hspace{1cm} (A-15)

which can be noted, is independent of the applied voltage \( E \) but merely a function of the circuit parameters.

Substituting equation A-15 in equation A-13 we have

\[ i_t = I_t e^{-(R/L)t} \]  \hspace{1cm} (A-16)
The value for $i_T$ is determined from initial conditions, i.e., the value of $i$ at $t = 0$. The total current $i = i_s + i_t$

$$i = E/R + I_T e^{-(R/L)t}$$

At $t = 0$

$$i = E/R + I_T = 0$$

whence $I_T = -E/R$.

The resultant expression for current is naturally the same as obtained by the classical solution:

$$i = E/R - E/R e^{-(R/L)t}$$

This example was for a system described by a first order differential equation. For the general system of $n$th order the transient component must be chosen in the form $I_n e^{pt}$. The values of $p_n$ are evaluated by setting the coefficient of $I_n e^{pt}$ equal to zero. These principles are covered by other more commonly used methods of differential equation solution such as those which use the LaPlace Transform method. We will not pursue the classical method of differential equation solution any further except to introduce the idea of the "characteristic equation" which is basic and which will also be derived with the LaPlace Transform method.

**Characteristic Equation**

The choice of the exponential form for the transient component of the solution of a linear set of differential equations was guided by the results of the classical solution.

This form of solution has the following interesting properties.

If

$$i = I e^{pt}$$

then

$$\frac{di}{dt} = Ipe^{pt}$$

and

$$\frac{d^2i}{dt^2} = Ip^2e^{pt}$$

Also

$$\int i \, dt = \frac{I}{p} e^{pt}$$

Hence in the equation for the transient solution, if $i$ is substituted by $Ie^{pt}$, the terms in the equation

$$\frac{d^n}{dt^n}$$

are replaced by $p^n$ and the terms $\int^n (\_) \, dt^n$ are replaced by $1/p^n$. For instance the differential equation
\[ \frac{a_0}{dt^n} \frac{d^n i}{dt^n} + \frac{a_1}{dt^{n-1}} \frac{d^{n-1} i}{dt^{n-1}} + \ldots + a_n i + a_{n+1} \int i dt = 0 \]

with \( i = Ie^{pt} \) becomes

\[ (a_0 p^n + a_1 p^{n-1} + \ldots + a_n + \frac{a_{n+1}}{p} + \ldots + \frac{a_{n+m}}{p^m}) Ie^{pt} = f(t) \]  \hspace{1cm} (A-22)

The polynomial form of the equation formed by substituting derivatives and integrals by the appropriate \( p \) and \( 1/p \) operators is called the operational form of the equation.

The basic equation which determines the transient modes is independent of the applied forcing function \( f(t) \). It is known as the system characteristic equation and in the example above is

\[ (a_0 p^n + a_1 p^{n-1} + \ldots + a_n + \frac{a_{n+1}}{p} + \ldots + \frac{a_{n+m}}{p^m}) = 0 \]

or

\[ (a_0 p^{n+m} + a_1 p^{n+m-1} + \ldots + a_n p^m + \ldots + a_{n+m}) = 0 \]  \hspace{1cm} (A-23)

The values of \( p \) which satisfy equation 23 are the roots of the characteristic equation and are the values that appear in the solution \( i = \int e^{pt} dt \) determining the transient modes of the system.

The characteristic equation of a system and its roots are fundamental to the evaluation of response and stability of dynamic systems.

**Example 1**

Fig. A-3 shows a series RLC network connected to a source \( E(t) \) by switch \( S \) at \( t = 0 \).

![Diagram of a series RLC network](attachment:image.png)

FIGURE A-3

The circuit equation for the time after closure of \( S \) is

\[ Ri + L \frac{di}{dt} + \frac{1}{C} \int i dt = E(t) \] \hspace{1cm} (A-24)
Breaking up the solution into two components (steady state, with same form as \( E(t) \) and transient) let us examine the case where \( E \) = constant.

The Steady State solution is found from equation A-24 by noting that \( i_s \) has the same form as \( E(t) \)

\[
\frac{di_s}{dt} = 0 \tag{A-25}
\]

Substituting equation A-25 in equation A-24

\[
R i_s + \frac{1}{C} \int i_s \, dt = E \tag{A-26}
\]

The only way that \( i_s \) can be a constant and satisfy equation A-26 is for \( i_s = 0 \) and \( \frac{1}{C} \int i_s \, dt = E \).

The Transient Solution is found by writing the left hand side of equation A-24 in operational form and setting it to zero.

\[
(Lp + R + \frac{1}{Cp}) i = 0 \tag{A-27}
\]

where \( i \) is of the form \( I_n e^{n t} \).

The characteristic equation of equation A-27 is

\[
CLp^2 + RCp + 1 = 0 \tag{A-28}
\]

which yields the roots

\[
P_1 = -\frac{R}{2L} + \frac{1}{2} \sqrt{\frac{R^2 - \frac{4}{LC}}{L}}
\]

and

\[
P_2 = -\frac{R}{2L} - \frac{1}{2} \sqrt{\frac{R^2 - \frac{4}{LC}}{L}}
\]

Depending on whether \((R/L)^2 > \) or \(< (4/LC)\) the roots \( P_1 \) and \( P_2 \) will be real or complex.

The expression for the transient current is

\[
i_t = I_1 e^{P_1 t} + I_2 e^{P_2 t} \tag{A-29}
\]

To evaluate \( I_1 \) and \( I_2 \), we note that the system's initial conditions were \( i = 0 \) and the voltage across the condenser = 0.

\[
i = 0 \tag{A-30}
\]

and

\[
\frac{1}{C} \int i \, dt = 0 \tag{A-31}
\]

Since the steady state component \( i_s = 0 \), condition equation A-30 applied to equation A-29 yields

\[
I_1 = -I_2 \tag{A-32}
\]
Also, applying equation A-30 and A-31 to equation A-24 at \( t = 0^+ \)

\[
\frac{d}{dt} \left( I_1 e^{p_1 t} + I_2 e^{p_2 t} \right) \bigg|_{t=0} = E
\]

i.e.

\[ I_1 p_1 + I_2 p_2 = \frac{E}{L} \]  \hspace{1cm} (A-33)

Solving equation A-32 and equation A-33

\[ I_1 = \frac{E}{L(p_1 - p_2)}, \quad I_2 = \frac{E}{L(p_2 - p_1)} \]  \hspace{1cm} (A-34)

And the total solution for \( i \) is

\[ i = \frac{E}{L} \left[ e^{p_1 t} - e^{p_2 t} \right] \]  \hspace{1cm} \frac{p_1 - p_2}{p_1 - p_2}  \hspace{1cm} (A-35)

For the case where \( p_1 \) and \( p_2 \) are complex conjugate roots \( \left( \frac{R}{L} \right)^2 < \frac{1}{LC} \),

i.e.: where

\[ p_1 = -\alpha + j\beta \]

and

\[ p_2 = -\alpha - j\beta \]  \hspace{1cm} (A-36)

Substituting these values in equation A-35

\[ i = \frac{E}{L} e^{-\alpha t} \left[ e^{j\beta t} - e^{-j\beta t} \right] \]  \hspace{1cm} (A-37)

which can be expressed as, from the definition of \( \sin \beta t \)

\[ i = \frac{E}{L} e^{-\alpha t} \sin \beta t \]  \hspace{1cm} (A-38)

Figure A-4 shows the nature of the current transient.
Example 2

Take the same example except that let the applied voltage be a sinusoidal function \( E = E \cos \omega t \) with the switch again closed at \( t = 0 \).

Again taking up the steady state solution, equation A-24 becomes

\[
Ri_s + L \frac{di_s}{dt} + \frac{1}{C} \int i_s \, dt = \frac{E}{2} (e^{+j\omega t} + e^{-j\omega t}) \tag{A-39}
\]

where

\[
e^{+j\omega t} + e^{-j\omega t} \over 2
\]

is the exponential form of \( \cos \omega t \).

Since \( i_s \) by definition will be of the same form as the applied voltage, we may further divide \( i_s \) into components corresponding to the applied voltage components

\[
i_{s+} = I_+ e^{+j\omega t} \tag{A-40}
\]

\[
i_{s-} = I_- e^{-j\omega t} \tag{A-41}
\]

where \( I_+ \) is the complex magnitude of \( i_{s+} \) and \( I_- \) is the complex magnitude of \( i_{s-} \). Considering these components individually, from equation A-39

\[
R \left[ I_+ e^{+j\omega t} \right] + L \frac{d}{dt} \left[ I_+ e^{+j\omega t} \right] + \frac{1}{C} \int I_+ e^{+j\omega t} \, dt = \frac{E}{2} e^{+j\omega t} + I_{s+} e^{+j\omega t} + \frac{I_+}{Cj\omega} e^{+j\omega t} = \frac{E}{2} e^{+j\omega t} \tag{A-42}
\]

Dividing both sides by \( e^{+j\omega t} \)

\[
R I_+ + L I_+ j\omega + \frac{I_+}{Cj\omega} = \frac{E}{2}
\]

or

\[
I_+ = \frac{E}{2 \left( R + j(\omega L - \frac{1}{C}) \right)} \tag{A-43}
\]

A similar derivation for \( I_- \) yields
\[
I_- = \frac{E}{2 \left( R - j(\omega L - \frac{1}{\omega C}) \right)} \quad (A-44)
\]

Equation A-43 can be expressed as

\[
I_+ = \frac{E \ e^{-j\theta_z}}{2 \ Z} \quad (A-45)
\]

where

\[
Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \quad (A-46)
\]

\[
\theta_z = \tan^{-1} \left( \frac{\omega L - \frac{1}{\omega C}}{R} \right) \quad (A-47)
\]

Likewise

\[
I_- = \frac{E \ e^{+j\theta_z}}{2 \ Z} \quad (A-48)
\]

The total steady state current \( i_s = i_{s_+} + i_{s_-} \). Using equation A-42 and equation A-43 and substituting equation A-47 and equation A-48,

\[
i_s = \frac{E}{Z} \left( e^{j(\omega t - \theta_z)} + e^{-j(\omega t - \theta_z)} \right) \quad (A-49)
\]

\[
i_s = \frac{E}{Z} \cos (\omega t - \theta_z)
\]

\( Z \) is the impedance of the network and indicates the ratio of voltage to current in the steady state for a sinusoidally varying applied voltage. Equation A-49 is of the same form as the applied voltage \( E \cos \omega t \). Its magnitude is \( E/Z \) and its phase angle with respect to the applied voltage sinusoid is \( \theta_z \).

The concept of operational impedance \( Z(p) = R + \frac{1}{Lp} + \frac{1}{Cp} \) is self-evident from equation A-27. By substituting \( p = j\omega \) one can derive the impedance to a fixed alternating voltage of frequency \( \omega \) rads/sec.

These concepts are important in the application of "frequency response" techniques which characterize the system in terms of its behavior as function of the frequency of the exciting function, \( \omega \).

Although the example was for an electric circuit, yielding the relationship between current and voltage the method is equally applicable to any variables of a system, be they mechanical, electrical or thermal, as long as they are related by linear differential equations.