Derivation of the Equations in Tables 6-7 through 6-11
In *Basic and Advanced Regulatory Control: System Design and Application*
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By Harold L. Wade
Figure 6-23. Block Diagram for an Ideal Process Model Liquid-Level Control Loop

Equation Number Legend

6-x = equations from Chapter 6  
D-x = derivation equations  
R-x = equations from the “Symbols and References” section of this document

Assume that the flow control loop is considerably faster than the level control loop (true in most practical situations), so that \( F(s) \) can be approximated by \( F(s) = 1 \).

Four transfer functions can be derived from the figure given above.

The set-point–to–level transfer function:

\[
\frac{L(s)}{SP(s)} = \frac{\frac{K_C}{T_L} s + \frac{K_C}{T_L}}{s^2 + \frac{K_C}{T_L} s + \frac{K_C}{T_L T_L}}
\]

Book (6-21); Derivation (D-1)
The set-point–to–flow-out transfer function:

$$\frac{F_{out}(s)}{SP(s)} = \frac{-K_C s \left( s + \frac{1}{T_I} \right)}{s^2 + \frac{K_C}{T_L} s + \frac{K_C}{T_I T_L}}$$

Book (6-22); Derivation (D-2)

The flow-in–to–level transfer function:

$$\frac{L(s)}{F_{in}(s)} = \frac{s}{T_L}$$

Derived below (D-3)

The flow-in–to–flow-out transfer function:

$$\frac{F_{out}(s)}{F_{in}(s)} = \frac{K_C s + K_C}{s^2 + \frac{K_C}{T_L} s + \frac{K_C}{T_I T_L}}$$

Book (6-30); Derivation (D-4)

Note that the set point–to–level and flow-in–to–flow-out transfer functions are identical.

From Figure 6-23, the transfer function $\frac{L(s)}{F_{in}(s)}$ can be derived:

$$L(s) = P(s)(F_{in}(s) - F_{out}(s))$$
$$F_{out}(s) = C(s)L(s)$$
$$L(s) = P(s)F_{in}(s) - P(s)C(s)L(s)$$
$$L(s) + P(s)C(s)L(s) = P(s)F_{in}(s)$$
$$(1 + P(s)C(s))L(s) = P(s)F_{in}(s)$$
$$\frac{L(s)}{F_{in}(s)} = \frac{P(s)}{1 + P(s)C(s)}$$

Derivation (D-5)
The elements of Equation D-5 are given by:

\[ P(s) = \frac{1}{T_L s} \]

\[ C(s) = K_C \left( 1 + \frac{1}{T_I s} \right) \]

Derivation (D-6)

Substitute Equation D-6 into Equation D-5. After simplification, this gives Equation D-3 above:

\[ L(s) = \frac{s}{s^2 + K_C \frac{T_L}{T_I T_L}} \]

\[ F_{in}(s) = \frac{T_L}{s^2 + K_C \frac{T_L}{T_I T_L}} \]

Notice that the denominators of Equations D-1, D-2, D-3, and D-4 are identical. These denominators can also be expressed in frequency response terms as:

\[ s^2 + K_C \frac{T_L}{T_I T_L} = s^2 + 2\zeta \omega_n s + \omega_n^2 \]
Determining PI Controller Tuning Parameters for Underdamped ($\zeta < 1$) Control Loop

Assuming that the tank holdup time, $T_L$, is known, the next step is to determine the desired loop decay ratio following a set-point change. Book Equation 6-25 then converts this to a damping factor, $\zeta$. The next step is to determine a maximum deviation of level from the set point, $\Delta L_{\text{max}}$, following an assumed worst-case step disturbance of inflow, $\Delta F_{\text{in}}$. (See Chapter 6, “Tuning for Integrating Processes,” subsection “Derived Equations for Liquid Level Control Tuning Parameters—Ideal Model” for further discussion of these parameters.)

Using these parameters and the relation for a step change in inflow, $F_{\text{in}}(s) = \frac{\Delta F_{\text{in}}}{s}$, Equation D-4 can be written and further developed as:

$$L(s) = \frac{\Delta F_{\text{in}}}{s T_L} \frac{s}{s^2 + \frac{K_C}{T_L} s + \frac{K_C}{T_I T_L}}$$

$$= \frac{\Delta F_{\text{in}}}{T_L} \frac{1}{s^2 + \frac{K_C}{T_L} s + \frac{K_C}{T_I T_L}}$$

$$= \frac{\Delta F_{\text{in}}}{T_L} \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

$$= \frac{\Delta F_{\text{in}}}{T_L \omega} \frac{\omega}{(s + \zeta \omega_n)^2 + \omega^2}$$

From the last equation, the time response of level to a step change in inflow is given by:

$$l(t) = \frac{\Delta F_{\text{in}}}{T_L \omega} e^{-\zeta \omega_n t} \sin(\omega t)$$

Derivation (D-7)
Extremities occur whenever \( \frac{dI(t)}{dt} = 0 \):

\[
\frac{dI(t)}{dt} = \frac{\Delta F_{in}}{T_L \omega} e^{-\zeta \omega_n t} (-\zeta \omega_n \sin(\omega t) + \omega \cos(\omega t))
\]

\[
\frac{dI(t)}{dt} = 0 \text{ whenever } \zeta \omega_n \sin(\omega t) = \omega \cos(\omega t)
\]

\[
\sin(\omega t) = \frac{\omega}{\zeta \omega_n} = \frac{\sqrt{1 - \zeta^2}}{\zeta}
\]

\[
\cos(\omega t) = \frac{1}{\zeta \omega_n} = \frac{1}{\zeta}
\]

\[
\tan(\omega t) = \frac{\sqrt{1 - \zeta^2}}{\zeta}
\]

The time of the first extremity is given by:

\[
t_{af} = t = \frac{1}{\omega} \tan^{-1} \left( \frac{\sqrt{1 - \zeta^2}}{\zeta} \right)
\]

Derivation (D-8)
Substitute \( \Delta a_F \) from Equation D-8 into Equation D-7 to find \( L_{\text{max}} \), the deviation of level at the first extremity.

\[
L_{\text{max}} = l(t_{a_F}) = \frac{\Delta F \omega_n}{T_L \omega} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \sin\left(\frac{1}{\omega} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)
\]

\[
L_{\text{max}} = l(t) = \frac{\Delta F \omega_n}{T_L \omega} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)} \left(\frac{1}{\sqrt{1-\zeta^2}}\right)
\]

\[
L_{\text{max}} = \frac{\Delta F \omega_n}{T_L \omega \sqrt{1-\zeta^2}} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}
\]

From R-3:

\[
T_L = \frac{K_C}{2 \zeta \omega_n}
\]

\[
\frac{L_{\text{max}}}{\Delta F} = \frac{1}{\frac{K_C}{2 \zeta \omega_n} \sqrt{1-\zeta^2}} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}
\]

\[
\frac{L_{\text{max}}}{\Delta F} = \frac{2 \zeta}{K_C \left(1-\zeta^2\right)} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}
\]
Solve for $K_C$:

$$K_C = \frac{2\zeta e^{\xi f(\zeta)}}{\left(\frac{L_{max}}{\Delta F_{in}}\right)(1-\zeta^2)}$$

Define where

$$f = f(\zeta)$$

$$K_C = \text{gain}$$

$$f(\zeta) = \frac{1}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$K_C = \frac{2\zeta e^{-\zeta f(\zeta)}}{(1-\zeta^2)\left(\frac{L_{max}}{\Delta F_{in}}\right)}$$

From R-4:

$$T_I = 4\zeta^2 \left(\frac{T_L}{K_C}\right)$$

From Equations D-10 and D-11:

$$T_I = 2\zeta(1-\zeta^2)T_L e^{\zeta f(\zeta)} \left(\frac{L_{max}}{\Delta F_{in}}\right)$$
Determining PI Controller Tuning Parameters for Underdamped ($\zeta = 1$) Control Loop

Note: With the ideal model, true critical damping ($\zeta = 1$) cannot be achieved because that would imply an infinite value for gain and zero value for integral time. Also, the period would be of infinite length. As a practical matter, $\zeta = 0.75$ would produce a decay ratio of 0.0008, and reasonable values for gain, integral time and period, and would be practically undistinguishable from 0 decay ratio.

Level Responses to Set-Point Change

Repeating Derivation Equation D-1, written in frequency response terms:

$$\frac{L(s)}{SP(s)} = \frac{\frac{K_C}{T_L}s + \frac{K_C}{T_I}T_L}{s^2 + \frac{K_C}{T_L}s + \frac{K_C}{T_I}T_L}$$

$$L(s) = \frac{K_C \Delta SP}{T_L} \frac{s + 1}{s\left(s^2 + 2\zeta \omega_n + \omega_n^2\right)}$$

By partial fraction expansion:

$$L(s) = \frac{K_C \Delta SP}{T_L} \left[ \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta \omega_n + \omega_n^2} \right]$$

Derivation (D-13)

Note that:

$$As^2 + 2\zeta \omega_n As + A\omega_n^2 + Bs^2 + Cs = s + \frac{1}{T_I}$$
Equating powers of $s$ gives:

$$A = \frac{1}{\omega_n^2 T_I}$$

$$B = -\frac{1}{\omega_n^2 T_I}$$

$$C = 1 - \frac{2\zeta \omega_n}{\omega_n^2 T_I} = 1 - \frac{2\zeta}{T_I} = 0$$

Substitute these values into Equation D-13 and simplify:

$$L(s) = \frac{K_C \Delta SP}{T_L} \left[ \frac{1}{\omega_n^2 T_I} - \frac{s}{s^2 + 2\zeta \omega_n + \omega_n^2} \right]$$

$$= \Delta SP \left[ \frac{1 - \frac{s}{(s + \zeta \omega_n)^2 + \omega_n^2}}{s} \right]$$

$$= \Delta SP \left[ \frac{1 - \frac{s + \zeta \omega_n - \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2}}{s} \right]$$

$$= \Delta SP \left[ \frac{1 + \frac{\zeta \omega_n}{\omega_n} \frac{\omega}{(s + \zeta \omega_n)^2 + \omega_n^2} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2}}{s} \right]$$

$$= \Delta SP \left[ \frac{1 + \frac{\zeta}{\sqrt{1-\zeta^2}} \frac{\omega}{(s + \zeta \omega_n)^2 + \omega_n^2} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2}}{s} \right]$$

Derivation (D-14)
From Equation D-14, the time response of level to a set-point change is given by:

\[
L(t) = \Delta SP \left[ 1 + e^{-\zeta \omega_t t} \left( \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega t) - \cos(\omega t) \right) \right]
\]

\[
= \Delta SP \left[ 1 + e^{-\zeta \omega_t t} \left( \frac{\sqrt{\zeta}}{\sqrt{1-\zeta^2}} + 1 \sin(\omega t - \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right)) \right) \right]
\]

\[
= \Delta SP \left[ 1 + e^{-\zeta \omega_t t} \left( \frac{1}{\sqrt{1-\zeta^2}} \sin(\omega t - \phi) \right) \right]
\]

Derivation (D-15)

where \( \phi = \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \)

Peaks and valleys of \( l(t) \) occur wherever the derivative is equal to 0.
From Equation D-15, the derivative is:

\[
\frac{d L(t)}{dt} = \Delta S P \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \left[ -\zeta \omega_n \sin(\omega t - \phi) + \omega \cos(\omega t - \phi) \right]
\]

\[
\frac{d L(t)}{dt} = 0 \text{ whenever}
\]

\[-\zeta \omega_n \sin(\omega t - \phi) + \omega \cos(\omega t - \phi) = 0\]

\[\zeta \omega_n \sin(\omega t - \phi) = \omega \cos(\omega t - \phi)\]

\[
\frac{\sin(\omega t - \phi)}{\cos(\omega t - \phi)} = \frac{\omega}{\zeta \omega_n}
\]

\[
= \frac{\omega_n \sqrt{1 - \zeta^2}}{\zeta \omega_n}
\]

\[
\tan(\omega t - \phi) = \frac{\sqrt{1 - \zeta^2}}{\zeta}
\]

\[
\omega t - \phi = \tan^{-1} \left( \frac{\sqrt{1 - \zeta^2}}{\zeta} \right)
\]

\[
\omega t - \phi = \phi_1
\]
Since \( \phi = \phi_i \) then \( \frac{d L(t)}{dt} = 0 \) whenever

\[
t = \frac{2 \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega}
\]

\[
t_{al} = t = \frac{T_i}{\zeta \sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)
\]

\[
t_{al} = t = \frac{T_i f(\zeta)}{\zeta}
\]

Derivation (D-17)

Book Table 6-9, Cell C3; Derivation (D-18)
Equation D-17 utilizes R-2 in the previous equation, and is written only for the first extremity. Equation D-18 utilizes Equation D-9 in Equation D-17, for reduction of algebraic symbols.

\[ L(t) \text{ at first peak} = L(t_{al}) \]. Substitute Equation D-17 into Equation D-15:

\[
L(t_{al}) = \Delta SP \left[ 1 + e^{-\zeta \omega_0 \tau_{l}(\zeta)} \left( \frac{1}{\sqrt{1-\zeta^2}} \sin \left( \omega - \frac{T_l}{\zeta \sqrt{1-\zeta^2}} \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} - \phi \right) \right) \right) \right]
\]

\[
= \sin \left( \frac{\omega}{\omega} \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} - \phi \right) \right)
\]

\[
= \sin \left( 2 \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) - \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \right)
\]

\[
= \sin \left( \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \right) - \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right)
\]

\[
= \sqrt{1-\zeta^2}
\]

\[
L(t_{al}) = \Delta SP \left[ 1 + e^{-\frac{2\zeta}{\sqrt{1-\zeta^2}} \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right)} \left( \frac{1}{\sqrt{1-\zeta^2}} \times \frac{1}{\sqrt{1-\zeta^2}} \times \sqrt{1-\zeta^2} \right) \right]
\]

Substitute \( f(\zeta) \) from Equation D-9 into the previous equation:

\[
\Delta L_{max} = L(t_{al}) = \Delta SP \left[ 1 + e^{-2\zeta f(\zeta)} \right]
\]

Book, Table 6-9, Cell C2
The overshoot ratio = \( \frac{L(t_{at}) - \Delta SP}{\Delta SP} = e^{-2\zeta f(\zeta)} \)

See R-8 for Book, Table 6-9, Cell B4

**Outflow Response to Step Inflow Change**

Since \( \frac{L(s)}{SP(s)} = \frac{F_{out}(s)}{F_{in}(s)} \) (see Equations D-1 and D-4), the first two items will require no new derivation will be required; only a change of symbols.

Maximum outflow change for step change in inflow (analogous to Equation D-19):

\[
\Delta F_{out_{\text{max}}} = \Delta F_{in} \left[ 1 + e^{-2\zeta f(\zeta)} \right]
\]

Book, Table 6-10, Cell B1

Outflow arrest time (analogous to level arrest time):

\[
t_{af} = t = \frac{T_f f(\zeta)}{\zeta}
\]

Book, Table 6-10, Cell B2; Derivation (D-18)
Maximum Rate of Change of Outflow

Analogous to Equation D-15:

\[ F_{\text{out}}(t) = \Delta F_{\text{in}} \left[ 1 + e^{-\zeta \omega_n t} \left( \frac{1}{\sqrt{1-\zeta^2}} \sin(\omega t - \phi) \right) \right] \]

where \( \phi = \tan^{-1}\left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \)

\[
\frac{d F_{\text{out}}(t)}{dt} = \Delta F_{\text{in}} \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \left[ -\zeta \omega_n \sin(\omega t - \phi) + \omega \cos(\omega t - \phi) \right]
\]

\[
= - \Delta F_{\text{in}} \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \left[ \sqrt{\zeta^2 \omega_n^2 + \omega^2} \sin(\omega t - \phi) \right]
\]

\[
\sqrt{\zeta^2 \omega_n^2 + \omega^2} = \sqrt{\zeta^2 \omega_n^2 + \omega_n^2 - \zeta^2 \omega_n^2} = \omega_n.
\]

\[ \phi_1 = \tan^{-1}\left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) = \phi \]

\[
\frac{d F_{\text{out}}(t)}{dt} = - \Delta F_{\text{in}} \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \sin(\omega t - 2\phi)
\]

From R-5:

\[ \omega_n = \frac{2 \zeta}{T_1} \]

Therefore:

\[
\frac{d F_{\text{out}}(t)}{dt} = - \Delta F_{\text{in}} \frac{2 \zeta}{T_1 \sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \sin(\omega t - 2\phi) \]

Derivation (D-21)
The maximum rate of change of $F_{out}$—that is, the maximum of $\frac{dF_{out}(t)}{dt}$—occurs where its derivative is $= 0$.

$$\frac{d^2 F_{out}(t)}{dt^2} = -\frac{\Delta F_{in}}{T_i} \frac{2\zeta}{\sqrt{1-\zeta^2}} \frac{d}{dt}\left[e^{-\zeta\omega_n t}\sin(\omega t - 2\phi)\right]$$

$$= -\frac{\Delta F_{in}}{T_i} \frac{2\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[-\zeta \omega_n \sin(\omega t - 2\phi) + \omega \cos(\omega t - 2\phi)\right]$$

$$= \frac{\Delta F_{in}}{T_i} \frac{2\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\zeta \omega_n \sin(\omega t - 2\phi) - \omega \cos(\omega t - 2\phi)\right]$$

$$= \frac{\Delta F_{in}}{T_i} \frac{2\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\sqrt{\omega^2 \omega_n^2 + \omega^2} \sin\left(\omega t - 2\phi - \tan^{-1}\left(\frac{\omega}{\zeta \omega_n}\right)\right)\right]$$

$$= \frac{\Delta F_{in}}{T_i} \frac{2\zeta \omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega t - 3\phi)$$

$$\frac{d^2 F_{out}(t)}{dt^2} = 0$$ whenever

$$\sin(\omega t - 3\phi) = 0$$

$$\omega t - 3\phi + n\pi = 0$$

$$t = \frac{3\phi - n\pi}{\omega}, \text{ for } n = \pm 0, 1, \ldots$$
Without derivation, but by observing chart records on the PC-ControLAB simulation, we will use \( n = -1 \). Hence \( \frac{dF_{\text{out}}(t)}{dt} \) achieves its maximum value at:

\[
t_{aR} = t = \frac{1}{\omega}(3\phi - \pi)
\]

Substitute \( t_{aR} \) from Equation D-22 into Equation D-21 to find the maximum rate of change of outflow:

\[
\frac{dF_{\text{out}}(t_{aR})}{dt} = -\frac{\Delta F_{\text{in}}}{T_I} \frac{2\zeta}{\sqrt{1-\zeta^2}} \left[ \exp \left( -\zeta\omega_n \left( \frac{1}{\omega}(3\phi - \pi) \right) \right) \right] \times \sin \left( \frac{\omega}{\omega}(3\phi - \pi) - 2\phi \right)
\]

\[
= -\frac{\Delta F_{\text{in}}}{T_I} \frac{2\zeta}{\sqrt{1-\zeta^2}} \left[ \exp \left( -3\zeta\omega_n \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) + \zeta\omega_n\pi \right) \right] \times \sin (\phi - \pi)
\]

Note that \( \sin (\phi - \pi) = -\sin (\phi) = -\sin \left( \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \right) = -\sqrt{1-\zeta^2} \).

Substituting this into the previous equation and simplifying yields:

\[
\frac{dF_{\text{out}}(t_{aR})}{dt} = \frac{\Delta F_{\text{in}}}{T_I} \times \frac{2\zeta}{\sqrt{1-\zeta^2}} \left( e^{-\zeta f(\phi)} \right)^3 \times \left( \frac{\zeta\pi}{e^{\sqrt{1-\zeta^2}}} \right)
\]

Book, Table 6-10, Row 5
Sinusoidal Fin: Amplitude Ratio, $F_{\text{out}}/F_{\text{in}}$

Let:

$$G(s) = \frac{F_{\text{out}}(s)}{F_{\text{in}}(s)} = \frac{\frac{K_C}{T_L} s + \frac{K_C}{T_I T_L}}{s^2 + \frac{K_C}{T_L} s + \frac{K_C}{T_I T_L}}$$

$$= \frac{\frac{K_C}{T_L} s + \frac{K_C}{T_I T_L}}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

$$= \frac{\frac{K_C}{T_L}}{\frac{1}{T_I} + \frac{s}{T_L}} \frac{1 + \frac{T_I s}{T_L}}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

$$G(j\omega) = \frac{\frac{K_C}{T_L}}{\frac{1}{T_I} + \frac{(j\omega)^2}{T_L}} \frac{1 + \frac{T_I (j\omega)}{\omega_n^2}}{s^2 + 2\zeta \omega_n (j\omega) + \omega_n^2}$$

$$= \frac{\frac{K_C}{T_L}}{\frac{1}{T_I} + \frac{j\omega T_I}{\omega_n^2}} \frac{1 + j\omega T_I}{\omega_n^2 - \omega^2} + j2\zeta \omega_n \omega$$

Book (6-30); Derivation (D-23)
Convert to standard complex number format (i.e., $A + jB$):

$$G(j\omega) = \frac{K_C}{T_j T_L} \frac{(1 + j\omega T_j)\left((\omega_n^2 - \omega^2) - j2\zeta\omega_n\omega\right)}{\left((\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega\right)\left((\omega_n^2 - \omega^2) - j2\zeta\omega_n\omega\right)}$$

$$= \omega_n^2 \left(\frac{(\omega_n^2 - \omega^2) + \omega T_j \left(2\zeta\omega_n\omega\right) + j(\omega T_j)(\omega_n^2 - \omega^2) - 2\zeta\omega_n\omega}{\left((\omega_n^2 - \omega^2)^2 + 2\zeta\omega_n\omega\right)^2}\right)$$

$$\text{Re}(G(j\omega)) = \omega_n^2 \left(\frac{(\omega_n^2 - \omega^2) + \omega T_j \left(2\zeta\omega_n\omega\right)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\right)$$

$$\text{Im}(G(j\omega)) = \omega_n^2 \left(\frac{(\omega T_j)(\omega_n^2 - \omega^2) - 2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\right)$$

$$G(j\omega) = \text{Re}(G(j\omega)) + j\text{Im}(G(j\omega))$$

$$\text{AR} = |G(j\omega)| = \frac{F_{out}(\omega)}{F_{in}(\omega)} = \sqrt{\left[\text{Re}(G(j\omega))\right]^2 + \left[\text{Im}(G(j\omega))\right]^2}$$

Book, Table 6-15, Row 15
Symbols and References

\[ K_C = \text{control gain, dimensionless} \]
\[ T_i = \text{integral time, minutes (per repeat)} \]
\[ T_L = \text{process time constant (tank time constant); see Equation 6-20} \]

\( \zeta \) Damping factor
\[ \zeta = 0 \text{ sustained oscillation} \]
\[ \zeta = 1 \text{ no oscillation} \]

DR Decay Ratio
\[ DR = 1 \text{ sustained oscillation} \]
\[ DR = 0 \text{ no oscillation} \]

Relations between \( \zeta \) and DR:
\[ \zeta = \frac{-\ln(DR)}{\sqrt{4\pi^2 + (\ln(DR))^2}} \]
\[ DR = \exp\left(\frac{-2\pi\zeta}{\sqrt{1 - \zeta^2}}\right) \]

\( \omega_n \) = undamped natural frequency
\( \omega \) = damped frequency, radians per minute (or other time units)
\[ \omega = \omega_n \sqrt{1 - \zeta^2} \quad \text{(R-1)} \]

Combining R-1 and R-5, given below, yields:
\[ \omega = \frac{2\zeta \sqrt{1 - \zeta^2}}{T_i} \quad \text{(R-2)} \]
From corresponding components of Equation 6-21 and 6-23:

\[ \frac{K_C}{T_L} = 2\zeta\omega_n \]  \hspace{1cm} (R-3)

\[ \frac{K_C}{T_I T_L} = \omega_n^2 \]  \hspace{1cm} (R-4)

Combining R-3 and R-4:

\[ \omega_n = \frac{2\zeta}{T_I} \]  \hspace{1cm} (R-5)

Combine R-1 and R-5 and solve for \( T_I \):

\[ T_I = \frac{4\zeta^2 T_L}{K_C} \]  \hspace{1cm} (R-6)

From R-5:

\[ \zeta = \frac{1}{2} \sqrt{\frac{K_C T_I}{T_L}} \]  \hspace{1cm} (R-7)

Period (utilizing R-2):

\[ P = \frac{2\pi}{\omega} = T_I \times \frac{\pi}{\zeta \sqrt{1-\zeta^2}} \]  \hspace{1cm} (R-8)